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Research Article

The Geometry of Invariant Submanifolds of a (κ,μ)-Paracontact Metric Manifold

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ABSTRACT

In this article, we consider an invariant submanifold of a (κ,μ)-paracontact metric manifold. We research the conditions Q(S,W_7·σ)=0, Q(S,W_3·σ)=0 and Q(g,W_7·σ)=0 for an invariant submanifold of a (κ,μ)-paracontact metric manifold. The results are significant and contribute the geometry of the (κ,μ)-paracontact metric manifold.

1. Introduction

Due to its vital applications in practical mathematics and science, the geometry of submanifolds has increased in prominence in modern differential geometry. In relativity theory, however, the idea of geodesics is crucial. For completely geodesic submanifolds, geodesics of the ambient manifolds are retained in the submanifolds.

Legendre foliations theory may be used to explain the geometry of paracontact metric manifolds. The characteristic vector field ξ-corresponds to the (κ,μ)-nullity condition for certain real constants κ and μ, hence it belongs to the class of paracontact manifolds [1].

V. Venkatesha and S. Basavarajappa studied the conformally flat, quasi conformally flat, and Weyl semisymmetric and it is shown that they are locally isometric to a sphere. Further many geometers researched the Lorentzian a-Sasakian manifolds with different curvature tensors and different connections [2].

U.C.De and S. Samui searched invariant submanifolds of Lorentzian para-Sasakian manifolds and some conditions that these submanifolds are totally geodesic. They gave necessary details about submanifolds and concircular curvature tensor and they studied about Lorentzian para-Sasakian manifolds and its submanifolds. Also, pseudoparallel and generalized Ricci pseudo-parallel invariant submanifolds of Lorentzian paraSasakian manifolds have been studied. In addition, they devoted to study invariant submanifolds satisfying the conditions $\tilde{Z}(\beta_1,\beta_2)\cdot a = fQ(g,a)$ and $\tilde{Z}(\beta_1,\beta_2)\cdot a = fQ(S,a)$ respectively [3]. Many geometers, inspired by these studies, studied invariant submanifolds of various manifolds [4, 5]

Since then several geometers studied curvature conditions and obtain important properties [6-19]

Recently, we have studied an invariant submanifold of a (κ,μ)-paracontact metric manifold. In this paper, we investigate the conditions $Q(S,W_7\cdot\sigma)=0$, $Q(S,W_3\cdot\sigma)=0$ and $Q(g,W_7\cdot\sigma)=0$ for an invariant submanifold of a (κ,μ) −paracontact metric manifold.

2. Preliminaries

A (2n+1)-dimensional smooth manifold $\tilde{M}$ is said to be a paracontact metric manifold if it admits a (1,1)-type tensor field ϕ, a unit vector field ξ, 1-form η and a semi-Riemannian metric tensor g which satisfy

$$\phi^2\beta_1 = \beta_1 - \eta(\beta_1)\xi,\quad \eta(\beta_1) = g(\beta_1,\xi) \tag{2.1}$$

$$g(\phi\beta_1,\phi\beta_2) = -g(\beta_1,\beta_2) + \eta(\beta_1)\eta(\beta_2),\quad \eta\circ\phi = 0, \tag{2.2}$$

and

$$d\eta(\beta_1,\beta_2) = g(\beta_1,\phi\beta_2),$$

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for all $\beta_1, \beta_2 \in \Gamma(T\widetilde{M})$, where $\Gamma(T\widetilde{M})$ denote the set of the differentiable vector fields on $\widetilde{M}$ [20].

In a paracontact metric manifold $(\widetilde{M}, \phi, \eta, \xi, g)$, we define a (1,1)-type tensor field by h. One can easily to see that h is a symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h \quad and \quad Trh = 0. \tag{2.3}$$

Moreover, for a (κ,μ) −paracontact metric manifold $\widetilde{M}$ of dimensional $(2n+1)$ and for all $\beta_1, \beta_2 \in \Gamma(T\widetilde{M})$, we have

$$(\widetilde{\nabla}_{\beta_1}\phi)\beta_2 = -g(\beta_1 - h\beta_1, \beta_2)\xi + \eta(\beta_2)(\beta_1 - h\beta_1), \tag{2.4}$$

where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. From (2.4), taking instead of ξ

$$\tilde{\nabla}_{\beta_1}\xi = -\phi\beta_1 + \phi h\beta_1, \tag{2.5}$$

for all $\beta_1 \in \Gamma(T\widetilde{M})$ [19].

A paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ is said to be a (κ,μ)-space form if its the Riemannian curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(\beta_1,\beta_2)\xi = \kappa\{\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2\}\mu\{\eta(\beta_2)h\beta_1 - \eta(\beta_1)h\beta_2\}, \tag{2.6}$$

for all $\beta_1, \beta_2 \in \Gamma(T\widetilde{M})$, where κ, μ are real constant [10]. The geometric structure of the (κ,μ)-paracontact metric manifold varies with $\kappa < -1$, $\kappa = -1$, and $\kappa > -1$. In addition, for the cases $\kappa < -1$ and $\kappa > -1$, (κ,μ)-nullity condition (2.6) entirely specifies the curvature tensor field [1].

In a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$, we have

$$S(\beta_1,\beta_2) = [2(1-n)+n\mu]g(\beta_1,\beta_2) + [2(n-1)+\mu]g(h\beta_1,\beta_2)$$
$$+[2(n-1)+n(2\kappa-\mu)]\eta(\beta_1)\eta(\beta_2), \tag{2.7}$$

$$S(\beta_1,\xi) = 2nk\eta(\beta_1), \quad Q\xi = 2nk\xi, \tag{2.8}$$

$$h^2 = (1+\kappa)\phi^2, \tag{2.9}$$

$$Q\phi - \phi Q = 2[2(n-1)+\mu]h\phi, \tag{2.10}$$

for all $\beta_1, \beta_2 \in \Gamma(T\widetilde{M})$, where S and Q denote the Ricci tensor and Ricci operator defined $S(\beta_1,\beta_2) = g(Q\beta_1,\beta_2)$.

On a semi-Riemannian manifold (M,g), for a (o,k)-type tensor field T and (0,2)-type tensor field A, $(0,k+2)$-type tensor field $Q(A,T)$ is defined as

$$Q(A,T)(\beta_{11},\beta_{12},\ldots,\beta_{1k};\beta_1,\beta_2) = -T((\beta_1 \wedge_A \beta_2)\beta_{11},\beta_{12},\ldots,\beta_{1k})$$
$$-T(\beta_{11},(\beta_1 \wedge_A \beta_2)\beta_1\beta_{13},\ldots,\beta_{1k})$$
$$-T(\beta_{11},\beta_{12},\ldots,(\beta_1 \wedge_A \beta_2)\beta_{1k}), \tag{2.11}$$

for all $\beta_{11},\beta_{12},\ldots,\beta_{1k},\beta_1,\beta_2 \in \Gamma(TM)$, where

$$(\beta_1 \wedge_A \beta_2)\beta_{11} = A(\beta_2,\beta_{11})\beta_1 - A(\beta_1,\beta_{11})\beta_2. \tag{2.12}$$

The W_3 −curvature tensor and W_7 −curvature tensor of a Riemannian manifold

$\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ are, respectively, given by

$$W_3(\beta_1,\beta_2)\beta_3 = R(\beta_1,\beta_2)\beta_3 - \tfrac{1}{2n}[S(\beta_1,\beta_3)\beta_2 - g(\beta_2,\beta_3)Q\beta_1] \tag{2.13}$$

and

$$W_7(\beta_1,\beta_2)\beta_3 = R(\beta_1,\beta_2)\beta_3 - \tfrac{1}{2n}[S(\beta_2,\beta_3)\beta_1 - g(\beta_2,\beta_3)Q\beta_1] \tag{2.14}$$

for all $\beta_1,\beta_2,\beta_3 \in \Gamma(T\widetilde{M})$ [21, 22]

3. Invariant Submanifolds of A (κ,μ)-Paracontact Metric Manifold

Now, let M be an immersed submanifold of a (κ,μ)-paracontact metric manifold

$\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$, by ∇ and $\nabla^\perp$, we denote the induced connections on $\Gamma(TM)$ and $\Gamma(T^\perp M)$, respectively. Then the Gauss and Weingarten formulas are, respectively, given by

$$\tilde{\nabla}_{\beta_1}\beta_2 = \nabla_{\beta_1}\beta_2 + \sigma(\beta_1,\beta_2) \tag{3.1}$$

and

$$\tilde{\nabla}_{\beta_1}\beta_5 = -A_{\beta_5}\beta_1 + \nabla^\perp_{\beta_1}\beta_5, \tag{3.2}$$

for all $\beta_1,\beta_2 \in \Gamma(TM)$ and $\beta_5 \in \Gamma(T^\perp M)$, where σ and A are called the second fundamental form and shape operator of M, respectively. They are related by

$$g(\sigma(\beta_1,\beta_2),\beta_5) = g(A_{\beta_5}\beta_1,\beta_2).$$

If $\widetilde{\nabla}_{\beta_1}\sigma = 0$, then the submanifold is said to be parallel of second fundamental form. The covariant derivatives of σ and A_{β_5} are defined by,

$$(\tilde{\nabla}_{\beta_1}\sigma)(\beta_2,\beta_3) = \nabla^\perp_{\beta_1}\sigma(\beta_2,\beta_3) - \sigma(\nabla_{\beta_1}\beta_2,\beta_3) - \sigma(\beta_2,\nabla_{\beta_1}\beta_3), \tag{3.3}$$

And

$$(\tilde{\nabla}_{\beta_1}A)_{\beta_5}\beta_2 = \nabla_{\beta_1}A_{\beta_5}\beta_2 - A_{\nabla^\perp_{\beta_1}\beta_5} - A_{\beta_5}\nabla_{\beta_1}\beta_2. \tag{3.4}$$

For an immersed submanifold M of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$, M is said to be invariant if the structure vector field ξ is tangent to M at every point of M and $\phi\beta_1$ is tangent to M for all $\beta_1 \in \Gamma(TM)$ at every point on M, that is, $\phi(T_{\beta_1}M) \subseteq T_{\beta_1}M$ at each point $\beta_1 \in M$. In the remainder of this work, we shall assume that M is an invariant submanifold unless otherwise stated.

Lemma 3.1 Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then the following relations hold.

$$\nabla_{\beta_1}\xi = -\phi\beta_1 + \phi h\beta_1 \tag{3.5}$$

$$\sigma(\beta_1,\xi) = 0, \tag{3.6}$$

$$\sigma(\phi\beta_1,\beta_2) = \sigma(\beta_1,\phi\beta_2) = \phi\sigma(\beta_1,\beta_2), \tag{3.7}$$

for all $\beta_1,\beta_2 \in \Gamma(TM)$.

Proof. Since the proof is a result of direct calculations.

Now, we will consider the curvature tensor W_3 of (κ,μ)-paracontact metric manifold form for later use. From (2.13) and (2.6), we have

$$W_3(\xi,\beta_2)\beta_3 = R(\xi,\beta_2)\beta_3 - \kappa(\eta(\beta_3)\beta_2 - g(\beta_2,\beta_3)\xi) =$$
$$2\kappa g(\beta_2,\beta_3)\xi - \eta(\beta_3)\beta_2] + \mu g(h\beta_2,\beta_3)\xi - \eta(\beta_3)h\beta_2]. \tag{3.8}$$

If we choose $\beta_1 = \xi$ in (2.14), we get

$$W_7(\xi,\beta_2)\beta_3 = R(\xi,\beta_2)\beta_3 - \tfrac{1}{2n}S(\beta_2,\beta_3)\xi + \kappa g(\beta_2,\beta_3)\xi = \kappa g(\beta_2,\beta_3)\xi -$$
$$\eta(\beta_3)\beta_2] + \mu g(h\beta_2,\beta_3)\xi - \eta(\beta_3)h\beta_2] - \tfrac{1}{2n}S(\beta_2,\beta_3)\xi + \kappa g(\beta_2,\beta_3)\xi. \tag{3.9}$$

Theorem 3.1 Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then $Q(S, W_7 \cdot \sigma) = 0$ if and only if M is either totally geodesic submanifold or

$$\kappa = \pm\sqrt{\mu^2(1+\kappa)} \quad provided \quad \kappa \neq 0.$$

Proof. From (2.11) and (2.12), we have

$$Q(S, W_7(\beta_1,\beta_2)\cdot\sigma)(\beta_4,\beta_5,\beta_6,\beta_3) = (W_7(\beta_1,\beta_2)\cdot$$
$$\sigma)((\beta_6 \wedge_S \beta_3)\beta_4,\beta_5) + (W_7(\beta_1,\beta_2)\cdot\sigma)(\beta_4,(\beta_6 \wedge_S \beta_3)\beta_5) = 0, \tag{3.10}$$

for any $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6 \in \Gamma(TM)$. This implies that

$$(W_7(\beta_1,\beta_2)\cdot\sigma)(S(\beta_3,\beta_4)\beta_6,\beta_5) - (W_7(\beta_1,\beta_2)\cdot\sigma)(S(\beta_6,\beta_4)\beta_3,\beta_5) +$$
$$(W_7(\beta_1,\beta_2)\cdot\sigma)(\beta_4,S(\beta_5,\beta_3)\beta_6) - (W_7(\beta_1,\beta_2)\cdot\sigma)(\beta_4,S(\beta_6,\beta_5)\beta_3) = 0 \tag{3.11}$$

This reduse for $\beta_2 = \beta_4 = \beta_5 = \beta_3 = \xi$, we have

$$(W_7(\beta_1,\xi)\cdot\sigma)(S(\xi,\xi)\beta_6,\xi) - (W_7(\beta_1,\xi)\cdot\sigma)(S(\beta_6,\xi)\xi,\xi) + (W_7(\beta_1,\xi)\cdot$$
$$\sigma)(\xi,S(\xi,\xi)\beta_6) - (W_7(\beta_1,\xi)\cdot\sigma)(\xi,S(\beta_6,\xi)\xi) = 0. \tag{3.12}$$

Also, by using (2.8), we arrive

$$2n\kappa(W_7(\beta_1,\xi)\cdot\sigma)(\beta_6,\xi) - 2n\kappa(W_7(\beta_1,\xi)\cdot\sigma)(\eta(\beta_6)\xi,\xi)$$
$$+2nk(W_7(\beta_1,\xi)\cdot\sigma)(\xi,\beta_6) - 2nk(W_7(\beta_1,\xi)\cdot\sigma)(\eta(\beta_6)\xi,\xi) = 0,$$

that is,

$$4n\kappa\{(W_7(\beta_1,\xi)\cdot\sigma)(\beta_6,\xi) - (W_7(\beta_1,\xi)\cdot\sigma)(\eta(\beta_6)\xi,\xi)\} = 0. \tag{3.13}$$

By virtue of (3.9) and (3.3), we arrive at

$$4n\kappa\{R^\perp(\beta_1,\xi)\sigma(\beta_6,\xi) - \sigma(W_7(\beta_1,\xi)\beta_6,\xi) - \sigma(\beta_6,W_7(\beta_1,\xi)\xi) -$$
$$R^\perp(\beta_1,\xi)\sigma(\eta(\beta_6)\xi,\xi) +$$
$$\sigma(\eta(\beta_6)W_7(\beta_1,\xi)\xi,\xi)\sigma(\eta(\beta_6)\xi,W_7(\beta_1,\xi)\xi)\} = 0. \tag{3.14}$$

Thus we obtain

$$4n\kappa\{\mu\sigma(\beta_6,h\beta_1) - \kappa\sigma(\beta_6,\beta_1)\} = 0. \tag{3.15}$$

If $h\beta_1$ is written instead of β_1 at (3.15) and using (2.9), we obtain

$$4n\kappa\{\mu\sigma(\beta_6,h^2\beta_1)-\kappa\sigma(\beta_6,h\beta_1)\}=4n\kappa\{\mu(1+\kappa)\sigma(\beta_6,\beta_1)-\kappa\sigma(\beta_6,h\beta_1)\}=0. \quad (3.16)$$

From (3.15) and (3.16), we concludee that

$$4n\kappa(\mu^2(1+\kappa)-\kappa^2)\sigma(\beta_6,\beta_1)=0.$$

This proves our assertion.

Theorem 3.2 Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then $Q(S,W_3\cdot\sigma)=0$ if and only if M is either totally geodesic submanifold or

$$\kappa=\pm\frac{1}{2}\sqrt{\mu^2(1+\kappa)},\quad provided\kappa\neq 0.$$

Proof. We suppose that $Q(S,W_3\cdot\sigma)=0$. This means that

$$Q(S,W_3(\beta_1,\beta_2)\cdot\sigma)(\beta_4,\beta_5,\beta_6,\beta_3)=(W_3(\beta_1,\beta_2)\cdot\sigma)((\beta_6\wedge_S\beta_3)\beta_4,\beta_5)+(W_3(\beta_1,\beta_2)\cdot\sigma)(\beta_4,(\beta_6\wedge_S\beta_3)\beta_5)=0, \quad (3.17)$$

for any $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\in\Gamma(TM)$. This implies that

$$(W_3(\beta_1,\beta_2)\cdot\sigma)(S(\beta_3,\beta_4)\beta_6,\beta_5)-(W_3(\beta_1,\beta_2)\cdot\sigma)(S(\beta_6,\beta_4)\beta_3,\beta_5)+(W_3(\beta_1,\beta_2)\cdot\sigma)(\beta_4,S(\beta_5,\beta_3)\beta_6)-(W_3(\beta_1,\beta_2)\cdot\sigma)(\beta_4,S(\beta_6,\beta_5)\beta_3)=0. \quad (3.18)$$

Again, putting $\beta_2=\beta_4=\beta_5=\beta_3=\xi$ in (3.18), we get

$$(W_3(\beta_1,\xi)\cdot\sigma)(S(\xi,\xi)\beta_6,\xi)-(W_3(\beta_1,\xi)\cdot\sigma)(S(\beta_6,\xi)\xi,\xi)+(W_3(\beta_1,\xi)\cdot\sigma)(\xi,S(\xi,\xi)\beta_6)-(W_3(\beta_1,\xi)\cdot\sigma)(\xi,S(\beta_6,\xi)\xi)=0 \quad (3.19)$$

In (3.19), by using (2.8), we reach at

$$2n\kappa(W_3(\beta_1,\xi)\cdot\sigma)(\beta_6,\xi)-2n\kappa(W_3(\beta_1,\xi)\cdot\sigma)(\eta(\beta_6)\xi,\xi)+2nk(W_3(\beta_1,\xi)\cdot\sigma)(\xi,\beta_6)-2nk(W_3(\beta_1,\xi)\cdot\sigma)(\eta(\beta_6)\xi,\xi)=0,$$

that is,

$$4n\kappa\{(W_3(\beta_1,\xi)\cdot\sigma)(\beta_6,\xi)-(W_3(\beta_1,\xi)\cdot\sigma)(\eta(\beta_6)\xi,\xi)\}=0. \quad (3.20)$$

Taking into account (3.3) and (3.8) in (3.20), we find

$$4n\kappa\{R^{\perp}(\beta_1,\xi)\sigma(\beta_6,\xi)-\sigma(W_3(\beta_1,\xi)\beta_6,\xi)-\sigma(\beta_6,W_3(\beta_1,\xi)\xi)-R^{\perp}(\beta_1,\xi)\sigma(\eta(\beta_6)\xi,\xi)+\sigma(\eta(\beta_6)W_3(\beta_1,\xi)\xi,\xi)+\sigma(\eta(\beta_6)\xi,W_3(\beta_1,\xi)\xi)\}=0. \quad (3.21)$$

After the necessary arrangements are made, we have

$$4n\kappa\{2\kappa\sigma(\beta_6,\beta_1)-\mu\sigma(\beta_6,h\beta_1)\}=0. \quad (3.22)$$

Taking $h\beta_1$ instead of β_1 in (3.22) and using (2.9), we obtain

$$4n\kappa\{2\kappa\sigma(\beta_6,h\beta_1)-\mu\sigma(\beta_6,h^2\beta_1)\}=4n\kappa\{2\kappa\sigma(\beta_6,h\beta_1)-(1+\kappa)\mu\sigma(\beta_6,h\beta_1)\}=0. \quad (3.23)$$

From (3.22) and (3.23), we conclude that

$$4n\kappa(4\kappa^2-\mu^2(1+\kappa))\sigma(\beta_6,\beta_1)=0,$$

which proves our assertion.

Theorem 3.3 Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. Then $Q(g,W_7\cdot\sigma)=0$ if and only if M is either totally geodesic submanifold or

$$\kappa=\pm\sqrt{\mu^2(1+\kappa)}.$$

Proof. Let us assume that $Q(g,W_7\cdot\sigma)=0$, that is,

$$Q(g,W_7(\beta_1,\beta_2)\cdot\sigma)(\beta_4,\beta_5,\beta_6,\beta_3)=(W_7(\beta_1,\beta_2)\cdot\sigma)\Big((\beta_3\wedge_g\beta_6)\beta_4,\beta_5\Big)+(W_7(\beta_1,\beta_2)\cdot\sigma)(\beta_4,(\beta_3\wedge_S\beta_6)\beta_5)=0 \quad (3.24)$$

for any $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\in\Gamma(TM)$. This implies that

$$(W_7(\beta_1,\beta_2)\cdot\sigma)(g(\beta_4,\beta_6)\beta_3,\beta_5)-(W_7(\beta_1,\beta_2)\cdot\sigma)(g(\beta_3,\beta_4)\beta_6,\beta_5)+(W_7(\beta_1,\beta_2)\cdot\sigma)(\beta_4,g(\beta_5,\beta_6)\beta_3)-(W_7(\beta_1,\beta_2)\cdot\sigma)(\beta_4,g(\beta_3,\beta_5)\beta_6)=0 \quad (3.25)$$

The relation (3.25) yields for $\beta_2=\beta_4=\beta_5=\beta_3=\xi$,

$$(W_7(\beta_1,\xi)\cdot\sigma)(\eta(\beta_6)\xi,\xi)-(W_7(\beta_1,\xi)\cdot\sigma)(\beta_6,\xi)+(W_7(\beta_1,\xi)\cdot\sigma)(\xi,\eta(\beta_6)\xi)-(W_7(\beta_1,\xi)\cdot\sigma)(\xi,\beta_6)=0, \quad (3.26)$$

that is,

$$(W_7(\beta_1,\xi)\cdot\sigma)(\eta(\beta_6)\xi,\xi)-2n\kappa(W_7(\beta_1,\xi)\cdot\sigma)(\beta_6,\xi)=0. \quad (3.27)$$

In view of (3.27), (3.3) and (3.9), we arrive at

$$\{R^{\perp}(\beta_1,\xi)\sigma(\eta(\beta_6)\xi,\xi)-\sigma(\eta(\beta_6)W_7(\beta_1,\xi)\xi,\xi)-\sigma(\eta(\beta_6)\xi,W_7(\beta_1,\xi)\xi)-R^{\perp}(\beta_1,\xi)\sigma(\beta_6,\xi)+\sigma(W_7(\beta_1,\xi)\beta_6,\xi)+\sigma(\beta_6,W_7(\beta_1,\xi)\xi)\}=0 \quad (3.28)$$

Here,

$$\mu\sigma(\beta_6,h\beta_1)+\kappa\sigma(\beta_6,\beta_1)=0. \quad (3.29)$$

Written $h\beta_1$ instead of β_1 in (3.29) and taking account of (2.9), we have

$$\mu\sigma(\beta_6,h^2\beta_1)+\kappa\sigma(\beta_6,h\beta_1)=\mu(1+\kappa)\sigma(\beta_6,\beta_1)+\kappa\sigma(\beta_6,h\beta_1)=0. \quad (3.30)$$

From (3.29) and (3.30), we conclude that

$$(\mu^2(1+\kappa)-\kappa^2)\sigma(\beta_6,\beta_1)=0.$$

The converse obvious. This completes the proof.

Conclusion 3.4 The results from this paper are as follows. Let be an (2n+1)-dimensional (κ,μ)-paracontact manifold. In this case,

$Q(S,W_7\cdot\sigma)=0$ if and only if M is either totally geodesic submanifold, or

$$\kappa=\pm\sqrt{\mu^2(1+\kappa)},\quad provided\quad \kappa\neq 0.$$

$Q(S,W_3\cdot\sigma)=0$ if and only if M is either totally geodesic submanifold , or

$$\kappa=\pm\frac{1}{2}\sqrt{\mu^2(1+\kappa)},\quad provided\kappa\neq 0.$$

$Q(g,W_7\cdot\sigma)=0$ if and only if M is either totally geodesic submanifold , or

$$\kappa=\pm\sqrt{\mu^2(1+\kappa)}.$$

Other manifolds can be run under these conditions.

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