Research Article

The Geometry of Invariant Submanifolds of a (κ,μ)-Paracontact Metric Manifold

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ABSTRACT

In this article, we consider an invariant submanifold of a (κ,μ)-paracontact metric manifold. We research the conditions Q(S,W_7⋅σ)=0, Q(S,W_3⋅σ)=0 and Q(g,W_7⋅σ)=0 for an invariant submanifold of a (κ,μ)-paracontact metric manifold. The results are significant and contribute the geometry of the (κ,μ)-paracontact metric manifold.

1. Introduction

Due to its vital applications in practical mathematics and science, the geometry of submanifolds has increased in prominence in modern differential geometry. In relativity theory, however, the idea of geodesics is crucial. For completely geodesic submanifolds, geodesics of the ambient manifolds are retained in the submanifolds.

Legendre foliations theory may be used to explain the geometry of paracontact metric manifolds. The characteristic vector field ξ-corresponds to the (κ,μ)-nullity condition for certain real constants κ and μ, hence it belongs to the class of paracontact manifolds [1].

V. Venkatesha and S. Basavarajappa studied the conformally flat, quasi conformally flat, and Weyl semi-symmetric and it is shown that they are locally isometric to a sphere. Further many geometers researched the Lorentzian para-Sasakian manifolds with different curvature tensors and different connections [2].

U.C.De and S. Samui searched invariant submanifolds of Lorentzian para-Sasakian manifolds and some conditions that these submanifolds are totally geodesic. They gave necessary details about submanifolds and concircular curvature tensor and they studied about Lorentzian para-Sasakian manifolds and its submanifolds. Also, pseudoparallel and generalized Ricci pseudo-parallel invariant submanifolds of Lorentzian para-Sasakian manifolds have been studied. In addition, they devoted to study invariant submanifolds satisfying the conditions $\tilde{Z}(\beta_1, \beta_2) \cdot a = fQ(g,a)$ and $\tilde{Z}(\beta_1, \beta_2) \cdot a = fQ(g,a)$ respectively [3]. Many geometers, inspired by these studies, studied invariant submanifolds of various manifolds [4, 5].

Since then several geometers studied curvature conditions and obtain important properties [6-19].

Recently, we have studied an invariant submanifold of a (κ,μ)-paracontact metric manifold. In this paper, we investigate the conditions $Q(S,W_3 \cdot \sigma)=0, Q(S,W_7 \cdot \sigma)=0$ and $Q(g,W_7 \cdot \sigma)=0$ for an invariant submanifold of a (κ,μ)-paracontact metric manifold.

2. Preliminaries

A (2n+1)-dimensional smooth manifold $\tilde{M}$ is said to be a paracontact metric manifold if it admits a (1,1)-type tensor field $\phi$, a unit vector field $\xi$, 1-form $\eta$ and a semi-Riemannian metric tensor $g$ which satisfy

$$\phi^2 \beta_i = \beta_i - \eta(\beta_i) \xi, \quad \eta(\beta_i) = g(\beta_i, \xi)$$  \hspace{1cm} (2.1)

$$g(\phi \beta_i, \phi \beta_j) = -g(\beta_i, \beta_j) + \eta(\beta_i) \eta(\beta_j), \quad \eta \star \phi = 0,$$  \hspace{1cm} (2.2)

and

$$d\eta(\beta_i, \beta_j) = g(\beta_i, \phi \beta_j).$$

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for all $\beta_1, \beta_2 \in \Gamma(TM)$, where $\Gamma(TM)$ denote the set of the differenciable vector fields on $M$ [20].

In a paracontact metric manifold $(\tilde{M}, \phi, \eta, \xi, g)$, we define a $(1,1)$-type tensor field by $\eta$. One can easily to see that $\eta$ is a symmetric and satisfies
\[ \eta^2 = 0, \quad \eta^\phi = -\phi \eta \text{ and } Tr\eta = 0. \]  

Moreover, for a $(\kappa, \mu)$-paracontact metric manifold $\tilde{M}$ of dimensional $(2n + 1)$ and for all $\beta_1, \beta_2 \in \Gamma(TM)$, we have
\[ (\nabla_\beta_2 \phi)_{\beta_1} = -\eta(\beta_1, \beta_2)\xi + \eta(\beta_2, \beta_1)\eta, \]  
where $\nabla$ denotes the Riemannian connection with respect to $g$.

From (2.4), taking instead of $\xi$
\[ \eta_{\beta_2, \beta_1} = -\phi \eta_{\beta_1} + \eta \phi \beta_1, \]  
for all $\beta_1, \beta_2 \in \Gamma(TM)$ [19].

A paracontact metric manifold $\tilde{M}^{2n+1}(\phi, \xi, g)$, is said to be $(\kappa, \mu)$-space form if its the Ricci curvature tensor $\tilde{R}$ satisfy
\[ \tilde{R}(\beta_1, \beta_2)\xi = \kappa n(\eta(\beta_1, \beta_2))\phi(\beta_1) + \mu(\eta(\beta_1, \beta_2))\eta(\beta_2), \]  
for all $\beta_1, \beta_2 \in \Gamma(TM)$, where $\kappa, \mu$ are real constant [10]. The geometric structure of the $(\kappa, \mu)$-paracontact metric manifold varies with $\kappa < -1, \mu = -1$, and $\kappa > -1$. In addition, for the cases $\kappa < -1$ and $\kappa > -1$, $(\kappa, \mu)$-nullity condition [26] entirely specifies the curvature tensor field [1].

In a $(\kappa, \mu)$-paracontact metric manifold $\tilde{M}^{2n+1}(\phi, \xi, g)$, we have
\[ S(\beta_1, \beta_2) = 2(1 - n) + \kappa n g(\beta_1, \beta_2) + 2(1 - 1) + \mu g(\eta(\beta_1, \beta_2), \eta(\beta_1, \beta_2)) + 2(2n - 1) + \mu(\kappa n - 1)g(\eta(\beta_1, \beta_2), \eta(\beta_2), \beta_1) = 0, \]  
for all $\beta_1, \beta_2 \in \Gamma(TM)$, where $S$ and $Q$ denote the Ricci tensor and Ricci operator defined
\[ S(\beta_1, \beta_2) = \kappa \eta(\beta_1, \beta_2) + (1 + 4)\eta(\beta_1, \beta_2) + 2(2n - 1) + \mu(\kappa n - 1)g(\eta(\beta_1, \beta_2), \eta(\beta_2), \beta_1), \]  
and
\[ \varphi(\beta_1, \beta_2) = \eta(\beta_1, \beta_2) = 0, \]  
for all $\beta_1, \beta_2 \in \Gamma(TM)$. In addition, for any $\beta_1, \beta_2, \beta_3 \in \Gamma(TM)$, we have
\[ S(\beta_1, \beta_2) + S(\beta_2, \beta_3) + S(\beta_3, \beta_1) = 0, \]  
for all $\beta_1, \beta_2, \beta_3 \in \Gamma(TM)$.

3. Invariant Submanifolds of A $(\kappa, \mu)$-Paracontact Metric Manifold

Now, let $M$ be an immersed submanifold of a $(\kappa, \mu)$-paracontact metric manifold $\tilde{M}^{2n+1}(\phi, \xi, g)$, by $\nabla$ and $\nabla^\perp$, we denote the induced connections on $\Gamma(TM)$ and $\Gamma(T^*M)$, respectively. Then the Gauss and Weingarten formulas are, respectively, given by
\[ \nabla_\beta_2 \phi = \nabla_\beta_2 + \sigma(\beta_1, \beta_2), \]  
and
\[ \nabla_\beta_2 \phi = \eta(\beta_1, \beta_2)\xi + \eta(\beta_2, \beta_1)\eta, \]  
for all $\beta_1, \beta_2 \in \Gamma(TM)$. In a paracontact metric manifold $\tilde{M}^{2n+1}(\phi, \xi, g, \sigma)$, we define a $(1,1)$-type tensor field by $\sigma$. One can easily to see that $\sigma$ is a symmetric and satisfies
\[ \sigma^2 = 0, \quad \sigma^\phi = -\phi \sigma \text{ and } Tr\sigma = 0. \]  
Moreover, for a $(\kappa, \mu)$-paracontact metric manifold $\tilde{M}$ of dimensional $(2n + 1)$ and for all $\beta_1, \beta_2 \in \Gamma(TM)$, we have
\[ (\nabla_\beta_2 \phi)_{\beta_1} = -\eta(\beta_1, \beta_2)\xi + \eta(\beta_2, \beta_1)\eta, \]  
where $\nabla$ denotes the Riemannian connection with respect to $g$.

From (2.4), taking instead of $\xi$
\[ \eta_{\beta_2, \beta_1} = -\phi \eta_{\beta_1} + \eta \phi \beta_1, \]  
for all $\beta_1, \beta_2 \in \Gamma(TM)$ [19].
obtain
\[ 4nK(\mu(\beta_\alpha h\beta_\alpha) - \sigma(\beta_\alpha h\beta_\alpha)) = 4nK(\mu(1 + \kappa)\sigma(\beta_\alpha h\beta_\alpha) - \sigma(\kappa(\beta_\alpha h\beta_\alpha))) = 0. \]  
(3.16)

From (3.15) and (3.16), we conclude that
\[ 4nK(\mu(1 + \kappa) - \kappa)\sigma(\beta_\alpha h\beta_\alpha) = 0. \]
This proves our assertion.

**Theorem 3.2** Let \( M \) be an invariant submanifold of a \((\kappa, \mu)\)-paracontact metric manifold \( M^{2n+1}(\varphi, \eta, \xi, g) \). Then \( Q(S, W_3) \cdot \sigma = 0 \) if and only if \( M \) is either totally geodesic submanifold or
\[ \kappa = \pm \sqrt{\mu(1 + \kappa)}, \text{ provided } \kappa \neq 0. \]

Proof. We suppose that \( Q(S, W_3) \cdot \sigma = 0 \). This means that
\[ Q(S, W_3) \cdot \sigma(\beta_\alpha h\beta_\alpha) = 0. \]
(3.17)

for any \( \beta_\alpha h\beta_\alpha, \beta_\alpha h\beta_\alpha, \beta_\alpha h\beta_\alpha \in \Gamma(TM) \). This implies that
\[ (W_3(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) - (W_3(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) = 0. \]
(3.18)

Again, putting \( \beta_3 = \beta_4 = \beta_5 = \xi \) in (3.18), we get
\[ (W_3(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) - (W_3(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) + (W_3(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) = 0. \]
(3.19)

In (3.19), by using (2.8), we reach at
\[ 4nK(W_3(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) - 2nK(W_3(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) + 2nK(W_3(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) = 0, \text{ that is,} \]
\[ 4nK(W_3(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) = 0. \]
(3.20)

Taking into account (3.3) and (3.8) in (3.20), we find
\[ 4nK(R^{+}(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) - \sigma(\beta_\alpha h\beta_\alpha) W_3(\beta_\alpha h\beta_\alpha) - R^{-}(\beta_\alpha h\beta_\alpha) \cdot \sigma(\beta_\alpha h\beta_\alpha) W_3(\beta_\alpha h\beta_\alpha) = 0. \]
(3.21)

After the necessary arrangements are made, we have
\[ 4nK(\mu(1 + \kappa) - \kappa)\sigma(\beta_\alpha h\beta_\alpha) = 0, \text{ which proves our assertion.} \]

**Theorem 3.3** Let \( M \) be an invariant submanifold of a \((\kappa, \mu)\)-paracontact metric manifold \( M^{2n+1}(\varphi, \eta, \xi, g) \). Then \( Q(g, W_7) \cdot \sigma = 0 \) if and only if \( M \) is either totally geodesic submanifold or
\[ \kappa = \pm \sqrt{\mu(1 + \kappa)}, \text{ provided } \kappa \neq 0. \]

Written \( h\beta_\alpha \) instead of \( \beta_\alpha \) in (3.29) and taking account of (2.9), we have
\[ \mu(\beta_\alpha h\beta_\alpha) + \sigma(\beta_\alpha h\beta_\alpha) = \mu(1 + \kappa)\sigma(\beta_\alpha h\beta_\alpha) + \sigma(\kappa(\beta_\alpha h\beta_\alpha)) = 0. \]
(3.29)

The converse obvious. This completes the proof.

**Conclusion 3.4** The results from this paper are as follows. Let be an \((2n+1)\)-dimensional \((\kappa, \mu)\)-paracontact manifold. In this case,
\[ Q(S, W_3) \cdot \sigma = 0 \text{ if and only if } M \text{ is either totally geodesic submanifold, or} \]
\[ \kappa = \pm \sqrt{\mu(1 + \kappa)}, \text{ provided } \kappa \neq 0. \]
\[ Q(g, W_7) \cdot \sigma = 0 \text{ if and only if } M \text{ is either totally geodesic submanifold, or} \]
\[ \kappa = \pm \sqrt{\mu(1 + \kappa)}. \]

Other manifolds can be run under these conditions.

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**References**


