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Research Article

Boundary Analysis of Control Systems Using Schwarz Lemma

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ABSTRACT

In this paper, Schwarz lemma at the boundary is considered for analysis of transfer functions used in control systems. Two theorems are presented with their proofs by performing boundary analysis of the derivative of positive real functions evaluated at the origin. Considering that the transfer function, $H(s)$, is an analytic function defined on the right half of the s- plane, inequalities for $|H'(0)|$ are obtained by assuming that is also analytic at the boundary point $s=0$ on the imaginary axis with $H(0)=0$. Finally, the sharpness of these inequalities is proved. As result of the sharpness analysis, different extremal functions corresponding to different transfer functions are obtained. The related block diagrams and root-locus curves are also presented for considered transfer functions. According to the root-locus diagrams, marginally stable transfer functions are obtained as the natural results of the theorems proposed in the study.

1. Introduction

Positive real functions (PRFs) play an important role in electrical engineering. Although they are mainly used in network synthesis as driving point impedance functions (DPIFs) [1, 2], it is also possible to encounter PRFs in signal processing [3], control systems [4], and even in electromagnetic and microwave engineering [5]. Positive realness for the systems is frequently investigated in control theory literature [6, 7].

In [8], it is aimed to find output feedback K to make the closed-loop system strictly positive real. It is also stated in the same study that the passivity is equivalent to positive realness for finite-dimensional linear time-invariant (LTI) systems.

As another application of positive realness in modern control theory, Kalman-Yakubovich-Popov (KYP) lemma (also known as the positive real lemma) can be given.

This lemma establishes the connection between the frequency domain, time domain, and state-space representation of the system [9].

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Accordingly, the derivative of the transfer function $H(s)$ is considered assuming that $H(s)$ is analytic at $s=0$ of the imaginary axis with $H(0)=0$. Performing sharpness analysis of obtained inequalities, unique transfer functions and related block diagrams with root-locus graphics are presented as the results of the study.

Before giving the preliminary considerations, the conditions for a transfer function to be qualified as positive real will be given.

A transfer function is said to be positive real if it satisfies the following conditions [10]:

i. $H(s)$ is analytic in $\Re s \geq 0$ except possibly for poles on the axis of imaginaries,

ii. $H(\overline{s})=\overline{H(s)}$

iii. $\Re H(s) \geq 0$, in $\Re s \geq 0$

The rest of the manuscript is organized as follows: In Section II, the preliminary considerations are given for the theorems to be discussed in the next section. In Section III, the main results and theorems are presented with explanatory examples and finally, conclusions are given in Section IV.

2. Preliminary Considerations

The well-known Schwarz's Lemma, which is a consequence of the Maximum Principle, says that if $f: D \rightarrow D$ is analytic with $f(z)=c_p z^p+c_{p+1} z^{p+1}+\cdots$. where $D=\{z:|z|<1$then

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$|f(z)| \leq |z|^p, \quad \forall z \in D$ *and consequently* $|c_p| \leq 1.$

Moreover, if the equality $|f(z)| = |z|^p$ holds for any $z \neq 0$, or $|c_p| = 1$ then f is a rotation, that is, $f(z) = z^p e^{i\theta}$, θ real [11].

Before applying Schwarz lemma, firstly, we will exploit the following map. Consider the product

$$B_0(z) = \prod_{k=1}^{n} \frac{z - b_k}{1 - \overline{b_k} z}.$$

The function $B_0(z)$ is called a finite Blaschke product, where $b_1, b_2, \ldots, b_n \in D$.

Let

$$f(z) = \frac{H(\frac{1+z}{1-z}) - H(1)}{H(\frac{1+z}{1-z}) + H(1)}, z = \frac{s-1}{s+1}, \quad (1.1)$$

where

$H(s) = H(1) + c_p(s-1)^p + c_{p+1}(s-1)^{p+1} + \cdots, p > 1.$

Note that $H(1)$ is real and positive. Here, $f(z)$ is an analytic function in D, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. Consider the function

$$\Phi(z) = \frac{f(z)}{\prod_{k=1}^{n} \frac{z - b_k}{1 - \overline{b_k} z}}, \quad b_k = \frac{s_k - 1}{s_k + 1}, k = 1,2,\ldots,n.$$

Here, $s1, s_2, \ldots, s_n$ are points in right half plane and $b_1, b_2, \ldots, b_n$ are zeros of $f(z)$. Also, $\Phi(z)$ is an analytic function in D, $\Phi(0) = 0$ and $|\Phi(z)| < 1$ for $z \in D$. Therefore, $\Phi(z)$ satisfies the conditions of the Schwarz lemma. Thus, from the Schwarz lemma, we obtain

$$\Phi(z) = \frac{f(z)}{\prod_{k=1}^{n} \frac{z - b_k}{1 - \overline{b_k} z}} = \frac{H(\frac{1+z}{1-z}) - H(1)}{H(\frac{1+z}{1-z}) + H(1)} \frac{1}{\prod_{k=1}^{n} \frac{z - b_k}{1 - \overline{b_k} z}}$$

$$= \frac{c_p \frac{2^p z^p}{(1-z)^p} + c_{p+1} \frac{2^{p+1} z^{p+1}}{(1-z)^{p+1}} +}{2H(1) + c_p \frac{2^p z^p}{(1-z)^p} + c_{p+1} \frac{2^{p+1} z^{p+1}}{(1-z)^{p+1}} + \prod_{k=1}^{n}} \frac{1}{\frac{z - b_k}{1 - \overline{b_k} z}},$$

$$\frac{\Phi(z)}{z^p} = \frac{c_p \frac{2^p}{(1-z)^p} + c_{p+1} \frac{2^{p+1} z}{(1-z)^{p+1}} + \ldots}{2H(1) + c_p \frac{2^p z^p}{(1-z)^p} + c_{p+1} \frac{2^{p+1} z^{p+1}}{(1-z)^{p+1}} + \ldots} \frac{1}{\prod_{k=1}^{n} \frac{z - b_k}{1 - \overline{b_k} z}}$$

$$|c_p| \leq \frac{H(1)}{2^{p-1}} \prod_{k=1}^{n} |b_k|$$

and

$$|c_p| \leq \frac{H(1)}{2^{p-1}} \prod_{k=1}^{n} \left| \frac{s_k - 1}{s_k + 1} \right|.$$

This result is sharp with equality for the function

$$H(s) = \left(-1 + \frac{2}{1 - \left(\frac{s-1}{s+1}\right)^p \prod_{k=1}^{n} \frac{\frac{s-1}{s+1} - \frac{s_k - 1}{s_k + 1}}{1 - \frac{s_k - 1}{s_k + 1} \frac{s-1}{s+1}}} \right) H(1),$$

where it can be simplified as follows:

$$H(s) = \frac{1 + \left(\frac{s-1}{s+1}\right)^p \prod_{k=1}^{n} \frac{s - s_k}{s + s_k}}{1 - \left(\frac{s-1}{s+1}\right)^p \prod_{k=1}^{n} \frac{s - s_k}{s + s_k}} H(1).$$

For different values of p and n, different transfer functions can be obtained. For simplicity, assume that $H(1) = 1$. Some examples are given below, respectively, for $p = 2$ with $n = 1$ and $p = 3$ with $n = 2$ cases:

$$H_{p=2,n=1}(s) = \frac{s^3 + (2s_1 + 1)s}{(2s_1 + 1)s^2 + s_1},$$

and

$$H_{p=2,n=3}(s) = \frac{b_1 s^5 + b_2 s^3 + b_3 s}{a_1 s^4 + a_2 s^2 + a_3},$$

where

$b_1 = 1,$

$b_2 = 2s_1 + 2s_2 + 2s_3 + s_1 s_2 + s_1 s_3 + s_2 s_3 + 1,$

$b_3 = s_1 s_2 + s_1 s_3 + s_2 s_3 + 2 s_1 s_2 s_3,$

$a_1 = s_1 + s_2 + s_3 + 2,$

$a_2 = 2s_1 s_2 + 2s_1 s_3 + 2s_2 s_3 + s_1 s_2 s_3 + s_1 + s_2 + s_3,$

$a_3 = s_1 s_2 s_3.$

For simplicity, assume that $s_1 = s_2 = s_3 = 1$. Then the transfer functions are given as

$$H_{p=2,n=1}(s) = \frac{s^3 + 3s}{3s^2 + 1},$$

and

$$H_{p=2,n=3}(s) = \frac{s^5 + 10s^3 + 5s}{5s^4 + 10s^2 + 1}.$$

The corresponding root-locus diagrams for $H_{p=2,n=1}(s)$ and $H_{p=2,n=3}(s)$ are given in Figs. 1 and 2, respectively. As it can be observed from the figures, the obtained transfer functions correspond to marginally stable systems.

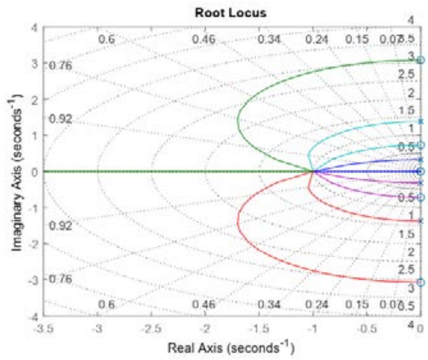

Fig. 1. Root-locus diagram for $H_{p=2,n=3}(s) = \frac{s^5+10s^3+5s}{5s^4+10s^2+1}$

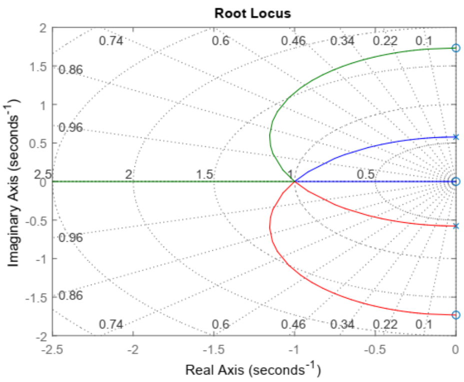

Fig 2: Root-locus diagram for $H_{p=2,n=1}(s) = \frac{s^3+3s}{3s^2+1}$.

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies is called the boundary version of Schwarz Lemma. An important result of Schwarz lemma was given by Osserman [12]. Also, it's still a hot topic in the mathematics literature [13-15].

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point c with $|c| = 1$, and if $|f(c)| = 1$ and $f^{'}(c)$ exists, then $\left|f^{'}(c)\right| \geq 1$, which is known as the Schwarz lemma on the boundary. In [12], R. Osserman proposed the boundary refinement of the classical Schwarz lemma as follows:

Let $f: D \to D$ be an analytic function with $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, $p \geq 1$. Assume that there is a $c \in \partial D$ so that f extends continuously to c, $|f(c)| = 1$ and $f^{'}(c)$ exists. Then

$$|f^{'}(c)| \geq p + \frac{1-|c_p|}{1+|c_p|} \tag{1.2}$$

and for $p = 1$,

$$|f^{'}(c)| \geq \frac{2}{1 + |f^{'}(0)|}$$

Inequality (1.2) is sharp, i. e., for $c = 1$, equality occurs for the function $f(z) = z^p \frac{z+\gamma}{1+\gamma z}$, $\gamma \in [0,1]$. Inequality (1.2) and its generalizations have important applications in geometric theory of functions, and they are still hot topics in the mathematics literature [13-15].

3. Main Results

In this section, boundary analysis results for the derivative of transfer function are presented. From the definition of PRFs, we can state that $H(s)$ is analytic and single valued on the right half of the s-plane. In Theorems 1 and 2 we establish lower bounds on the derivative of $H(s)$ for positive real functions with $H(0) = 0$.

Theorem 2.1 *Let $H(s) = H(1) + c_p(s-1)^p + c_{p+1}(s-1)^{p+1} + \cdots$, $p \geq 2$ be a positive real function that is also analytic at the point $s = 0$ of the imaginary axis with $H(0) = 0$. Then*

$$|H^{'}(0)| \geq H(1)\left(p + \frac{2\left(H(1) - 2^{p-1}|c_p|\right)^2}{(H(1))^2 - \left(2^{p-1}|c_p|\right)^2 + 2^{p-1}H(1)|pc_p + 2c_{p+1}|}\right). \tag{2.1}$$

The equality in (2.1) occurs for the function

$$H(s) = \frac{(s+1)^{p+1} + (s-1)^{p+1}}{(s+1)^{p+1} - (s-1)^{p+1}} H(1), \quad p = 2,4,6,\ldots,n.$$

Proof. Let $r(z) = z^p$, $z \in D$ and $f(z)$ be the same as in (1.1). $r(z)$ is analytic in D and $|r(z)| < 1$ for $|z| < 1$. The maximum principle implies that for each $z \in D$, we have $|f(z)| \leq |r(z)|$. Thus,

$$m(z) = \frac{f(z)}{r(z)}$$

is an analytic function in D and $|m(z)| < 1$ for $|z| < 1$. In particular, using Schwarz lemma, we take

$$m(z) = \frac{f(z)}{r(z)} = \frac{H(\frac{1+z}{1-z}) - H(1)}{\left[H(\frac{1+z}{1-z}) + H(1)\right] z^p} = \frac{c_p \frac{2^p z^p}{(1-z)^p} + c_{p+1} \frac{2^{p+1} z^{p+1}}{(1-z)^{p+1}} + \ldots}{\left[2H(1) + c_p \frac{2^p z^p}{(1-z)^p} + c_{p+1} \frac{2^{p+1} z^{p+1}}{(1-z)^{p+1}} + \ldots\right] z^p}$$

$$= \frac{c_p \frac{2^p}{(1-z)^p} + c_{p+1} \frac{2^{p+1}}{(1-z)^{p+1}} z + \ldots}{2H(1) + c_p \frac{2^p z^p}{(1-z)^p} + c_{p+1} \frac{2^{p+1} z^{p+1}}{(1-z)^{p+1}} + \ldots},$$

$$|m(0)| = \frac{2^{p-1}}{H(1)} |c_p| \leq 1$$

and

$$|m^{'}(0)| = \frac{2^{p-1}}{H(1)} |pc_p + 2c_{p+1}|.$$

If $|m(0)| = 1$ then by the maximum principle, we have $\frac{f(z)}{r(z)} = e^{i\theta}$, $\theta \in \mathscr{R}$, $f(z) = e^{i\theta} r(z) = z^p e^{i\theta}$ and

$$H\left(\frac{1+z}{1-z}\right) = \frac{1 + z^p e^{i\theta}}{1 - z^p e^{i\theta}}.$$

Further we may assume

$$H\left(\frac{1+z}{1-z}\right) \neq \frac{1 + z^p e^{i\theta}}{1 - z^p e^{i\theta}},$$

and thus $|m(0)| < 1$.

Therefore, we take $|m(0)| = \frac{2^{p-1}}{H(1)} |c_p| < 1$.

In addition, since the expression $\frac{cf^{'}(c)}{f(c)}$ is a real number greater than or equal to 1 [16] and $H(0) = 0$ yields $|f(c)| = 1$, $c = -1 \in \partial D$, we take

$$\frac{cf^{'}(c)}{f(c)} = \left|\frac{cf^{'}(c)}{f(c)}\right| = |f^{'}(c)|.$$

Also, since $|f(z)| \leq |r(z)|$, we get

$$\frac{1 - |f(z)|}{1 - |z|} \geq \frac{1 - |r(z)|}{1 - |z|}.$$

Without loss of generality, passing to limit in the last inequality yields

$$|f^{'}(c)| \geq |r^{'}(c)|.$$

Thus, we obtain

$$\frac{cf^{'}(c)}{f(c)} = |f^{'}(c)| \geq |r^{'}(b)| = \frac{cr^{'}(c)}{r(c)}, \qquad c \in \partial D.$$

The composite function

$$\Theta(z) = \frac{m(z) - m(0)}{1 - \overline{m(0)} m(z)}$$

satisfies the hypothesis of the Schwarz lemma on the boundary as shown below:

First, let us show that $|\Theta(z)| < 1$ for $z \in D$. Since

$$|m(z) - m(0)|^2 = |m(z)|^2 - m(z)\overline{m(0)} - m(0)\overline{m(z)} + |m(0)|^2$$

and

$$\left|1 - \overline{m(0)} m(z)\right|^2 = 1 - m(0)\overline{m(z)} - m(z)\overline{m(0)} + |m(0)|^2 |m(z)|^2,$$

then

$$|m(z) - m(0)|^2 - \left|1 - \overline{m(0)} m(z)\right|^2 = -(1 - |m(0)|^2)(1 - |m(z)|^2) < 0.$$

Therefore, we take

$$|m(z) - m(0)|^2 < \left|1 - \overline{m(0)} m(z)\right|^2$$

and

$$|\Theta(z)| < 1.$$

Second, we will prove that $|\Theta(-1)| = 1$ for $c = -1 \in \partial D$. Since

$$\Theta(z) = \frac{m(z) - m(0)}{1 - \overline{m(0)} m(z)} = \frac{m(z) - m(0)}{m(z)\left(\frac{\overline{m(z)}}{|m(z)|^2} - m(0)\right)}$$

and

$$m(z) = \frac{f(z)}{r(z)} = \frac{H\left(\frac{1+z}{1-z}\right) - H(1)}{\left(H\left(\frac{1+z}{1-z}\right) + H(1)\right) z^p},$$

then for $c = -1 \in \partial D$ and $H(0) = 0$ we take

$$|m(-1)| = 1.$$

In this way, we get

$$|\Theta(-1)| = 1.$$

Thus, from (1.3), we obtain the estimate

$$\frac{2}{1 + |\Theta^{'}(0)|} \leq |\Theta^{'}(-1)| = \frac{1 - |m(0)|^2}{\left|1 - \overline{m(0)} m(-1)\right|^2} |m^{'}(-1)|$$

$$\leq \frac{1 + |m(0)|}{1 - |m(0)|} \left|\frac{f^{'}(-1)}{r(-1)} - \frac{f(-1) r^{'}(-1)}{r^2(-1)}\right|$$

$$= \frac{1 + |m(0)|}{1 - |m(0)|} \left|\frac{f(-1)}{(-1) r(-1)}\right| \left|\frac{(-1) f^{'}(-1)}{f(-1)} - \frac{(-1) r^{'}(-1)}{r(-1)}\right|$$

$$= \frac{1 + |m(0)|}{1 - |m(0)|} (|f^{'}(-1)| - |r^{'}(-1)|) = \frac{1 + |m(0)|}{1 - |m(0)|} (|f^{'}(-1)| - p).$$

Since

$$\Theta^{'}(z) = \frac{1 - |m(0)|^2}{\left(1 - \overline{m(0)} \varphi(z)\right)^2} m^{'}(z),$$

$$|\Theta'(0)| = \frac{|m'(0)|}{1-|m(0)|^2} = \frac{\frac{2^{p-1}|pc_p+2c_{p+1}|}{H(1)}}{1-\left(\frac{2^{p-1}|c_p|}{H(1)}\right)^2}$$

and

$$|f'(-1)| = \frac{|H'(0)|}{H(1)},$$

we obtain

$$\frac{2}{1+H(1)\frac{2^{p-1}|pc_p+2c_{p+1}|}{(H(1))^2-\left(2^{p-1}|c_p|\right)^2}} \leq \frac{H(1)+2^{p-1}|c_p|}{H(1)-2^{p-1}|c_p|}\left(\frac{|H'(0)|}{H(1)}-p\right),$$

$$\frac{2\left((H(1))^2-\left(2^{p-1}|c_p|\right)^2\right)}{(H(1))^2-\left(2^{p-1}|c_p|\right)^2+2^{p-1}H(1)|pc_p+2c_{p+1}|}\frac{H(1)-2^{p-1}|c_p|}{H(1)+2^{p-1}|c_p|}+p \leq \frac{|H'(0)|}{H(1)}$$

and

$$|H'(0)| \geq H(1)\left(p+\frac{2\left(H(1)-2^{p-1}|c_p|\right)^2}{(H(1))^2-\left(2^{p-1}|c_p|\right)^2+2^{p-1}H(1)|pc_p+2c_{p+1}|}\right).$$

Therefore, we get the inequality (2.1).

Now, we shall show that the inequality (2.1) is sharp. Let

$$H(s) = \frac{(s+1)^{p+1}+(s-1)^{p+1}}{(s+1)^{p+1}-(s-1)^{p+1}}H(1).$$

Then

$$H'(s) = \frac{4(p+1)(s-1)^p(s+1)^p}{((s-1)^{p+1}-(s+1)^{p+1})^2}H(1)$$

and

$$H'(0) = \frac{4(p+1)}{|(-1)^{p+1}-1|^2}H(1).$$

Therefore, for $p = 2,4,6,\ldots,n$, we obtain

$$|H'(0)| = (p+1)H(1).$$

On the other hand, we obtain

$$H(1)+c_p(s-1)^p+c_{p+1}(s-1)^{p+1}+\ldots = \frac{(s+1)^{p+1}+(s-1)^{p+1}}{(s+1)^{p+1}-(s-1)^{p+1}}H(1),$$

$$c_p(s-1)^p+c_{p+1}(s-1)^{p+1}+\ldots = \left(\frac{(s+1)^{p+1}+(s-1)^{p+1}}{(s+1)^{p+1}-(s-1)^{p+1}}-1\right)H(1),$$

$$c_p(s-1)^p+c_{p+1}(s-1)^{p+1}+\ldots = 2\frac{(s-1)^{p+1}}{(s+1)^{p+1}-(s-1)^{p+1}}H(1)$$

and

$$c_p+c_{p+1}(s-1)+\ldots = 2\frac{(s-1)}{(s+1)^{p+1}+(s-1)^{p+1}}H(1).$$

Passing to limit in the last equality yields $c_p = 0$. Similarly, using straightforward calculations, we take $c_{p+1} = \frac{1}{2^p}H(1)$. So, we take

$$H(1)\left(p+\frac{2\left(H(1)-2^{p-1}|c_p|\right)^2}{(H(1))^2-\left(2^{p-1}|c_p|\right)^2+2^{p-1}H(1)|pc_p+2c_{p+1}|}\right) = (p+1)H(1).$$

The extremal function can be considered as a transfer function of a certain control system. For simplicity, assume that $H(1) = 1$. Then, the extremal function obtained in Theorem 1 can be rewritten as

$$H(s) = \frac{(s+1)^{p+1}+(s-1)^{p+1}}{(s+1)^{p+1}-(s-1)^{p+1}}, \quad p = 2,4,6,\ldots,n.$$

This transfer function can be implemented as a block as given in Figure 3.

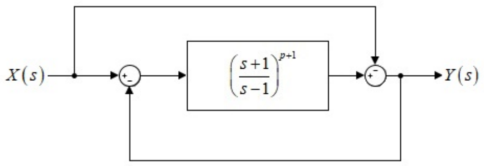

Fig. 3: Block diagram representation of the transfer function $H(s) = \frac{(s+1)^{p+1}+(s-1)^{p+1}}{(s+1)^{p+1}-(s-1)^{p+1}}, \quad p = 2,4,6,\ldots,n.$

As exemplary applications, $p = 2$ and $p = 4$ cases have been considered. The corresponding transfer functions are given respectively as follows:

$$H_{p=2}(s) = \frac{s^3+3s}{3s^2+1}$$

$$H_{p=4}(s) = \frac{s^5+10s^3+5s}{5s^4+10s^2+1},$$

where related root locus diagrams are given in Figs. 4 and 5, respectively. As it can be seen from the figures, both transfer functions belong to marginally stable systems as all the poles are located on the imaginary axis.

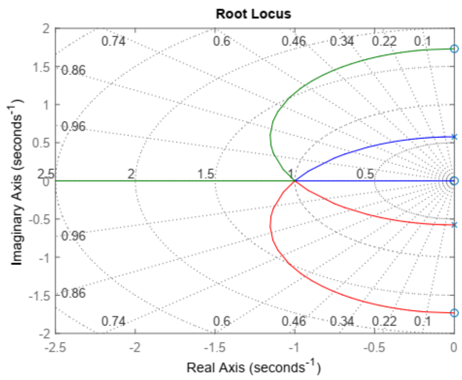

Fig. 4. Root-locus diagram for $H_{p=2}(s) = \frac{s^3+3s}{3s^2+1}$.

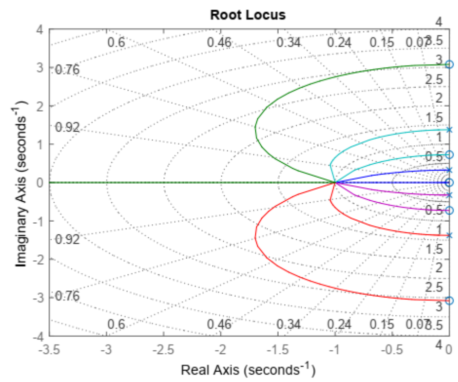

Fig. 5: Root-locus diagram for $H_{p=4}(s) = \frac{s^5+10s^3+5s}{5s^4+10s^2+1}$.

Theorem 2.2 *Let $H(s) = H(1)+c_p(s-1)^p+c_{p+1}(s-1)^{p+1}+\ldots$, $p \geq 2$ be a positive real function that is also an analytic at the point $s = 0$ of the imaginary axis with $H(0) = 0$. Assume that $s_1, s_2, \ldots, s_n$ are points in the right half plane that are different from one with $H(s_k) = H(1)$, $k = 1,2,\ldots,n$. Then we have the inequality (2.2).*

$$|H'(0)| \geq H(1)\left(p+\sum_{k=1}^{n}\frac{\Re s_k}{|s_k|^2}\right. \quad (2.2)$$

$$\left.+\frac{2\left(H(1)\prod_{k=1}^{n}\left|\frac{s_k-1}{s_k+1}\right|-\left(2^{p-1}|c_p|\right)\right)^2}{\left(H(1)\prod_{k=1}^{n}\left|\frac{s_k-1}{s_k+1}\right|\right)^2-\left(2^{p-1}|c_p|\right)^2+2^{p-1}H(1)\prod_{k=1}^{n}\left|\frac{s_k-1}{s_k+1}\right|\left|2c_{p+1}+c_p\left(p+\sum_{k=1}^{n}\frac{4\Re s_k}{|s_k|^2+2i\Im s_k-1}\right)\right|}\right) \quad (2.2)$$

The result (2.2) is sharp for the function given by

$$H(s)=\frac{1-\left(\frac{s-1}{s+1}\right)^{p+1}\prod_{k=1}^{n}\frac{\frac{s-1}{s+1}-\frac{s_k-1}{s_k+1}}{1-\overline{\frac{s_k-1}{s_k+1}}\frac{s-1}{s+1}}}{1+\left(\frac{s-1}{s+1}\right)^{p+1}\prod_{k=1}^{n}\frac{\frac{s-1}{s+1}-\frac{s_k-1}{s_k+1}}{1-\overline{\frac{s_k-1}{s_k+1}}\frac{s-1}{s+1}}}H(1),\quad p=2,4,6,\ldots,$$

where $s_1,s_2,\ldots,s_n$ are positive real numbers.

Proof. Let $f(z)$ be as in the proof of Theorem 1 and $b_1,b_2,\ldots,b_n$ be the zeros of the function $f(z)$ in D that are different from zero.

The function

$$B(z)=z^p\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}$$

is analytic in D, and $|B(z)|<1$ for $z\in D$. By the maximum principle, for each $z\in D$, we have $|f(z)|\leq|B(z)|$.

Let

$$h(z)=\frac{f(z)}{B(z)}.$$

This function is analytic in D and $|h(z)|\leq 1$ for $z\in D$.

Therefore, we have

$$h(z)=\frac{c_p\frac{2^pz^p}{(1-z)^p}+c_{p+1}\frac{2^{p+1}z^{p+1}}{(1-z)^{p+1}}+\ldots}{\left[2H(1)+c_p\frac{2^pz^p}{(1-z)^p}+c_{p+1}\frac{2^{p+1}z^{p+1}}{(1-z)^{p+1}}+\ldots\right]z^p\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}\qquad(1)$$

$$=\frac{c_p\frac{2^p}{(1-z)^p}+c_{p+1}\frac{2^{p+1}}{(1-z)^{p+1}}z+\ldots}{\left(2H(1)+c_p\frac{2^pz^p}{(1-z)^p}+c_{p+1}\frac{2^{p+1}z^{p+1}}{(1-z)^{p+1}}+\ldots\right)\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}},$$

$$|h(0)|=\frac{2^{p-1}|c_p|}{H(1)\prod_{k=1}^{n}|b_k|}$$

and

$$|h'(0)|=\frac{2^{p-1}\left|2c_{p+1}+c_p\left(p+\sum_{k=1}^{n}\frac{1-|b_k|^2}{b_k}\right)\right|}{H(1)\prod_{k=1}^{n}|b_k|}.$$

Moreover, it is obvious that

$$|B'(c)|=\frac{cB'(c)}{B(c)}=p+\sum_{k=1}^{n}\frac{1-|b_k|^2}{|c-b_k|^2}$$

and

$$\frac{cf'(c)}{f(c)}=|f'(c)|\geq|B'(c)|=\frac{cB'(c)}{B(c)},\qquad c\in\partial D.$$

Consider the auxiliary function

$$\Upsilon(z)=\frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}.$$

This function is analytic in the unit disc D, $\Upsilon(0)=0$, $|\Upsilon(z)|<1$ for $|z|<1$ and $|\Upsilon(c)|=1$ for $-1=c\in\partial D$. Therefore, from Schwarz lemma on the boundary for $p=1$, we obtain

$$\frac{2}{1+|\Upsilon'(0)|}\leq|\Upsilon'(-1)|=\frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(-1)\right|^2}|h'(-1)|$$

$$\leq\frac{1+|h(0)|}{1-|h(0)|}\{|f'(-1)|-|B'(-1)|\}.$$

Also, it can be seen that

$$\Upsilon'(z)=\frac{1-|h(0)|^2}{\left(1-\overline{h(0)}h(z)\right)^2}h'(z)$$

and

$$\Upsilon'(z)=\frac{|h'(0)|}{1-|h(0)|^2}=\frac{\frac{2^{p-1}\left|2c_{p+1}+c_p\left(p+\sum_{k=1}^{n}\frac{1-|b_k|^2}{b_k}\right)\right|}{H(1)\prod_{k=1}^{n}|b_k|}}{1-\left(\frac{2^{p-1}|c_p|}{H(1)\prod_{k=1}^{n}|b_k|}\right)^2}$$

$$=2^{p-1}H(1)\prod_{k=1}^{n}|b_k|\frac{\left|2c_{p+1}+c_p\left(p+\sum_{k=1}^{n}\frac{1-|b_k|^2}{b_k}\right)\right|}{\left(H(1)\prod_{k=1}^{n}|b_k|\right)^2-\left(2^{p-1}|c_p|\right)^2}.$$

Thus, we take

$$\frac{2}{1+2^{p-1}H(1)\prod_{k=1}^{n}|b_k|\frac{\left|2c_{p+1}+c_p\left(p+\sum_{k=1}^{n}\frac{1-|b_k|^2}{b_k}\right)\right|}{\left(H(1)\prod_{k=1}^{n}|b_k|\right)^2-\left(2^{p-1}|c_p|\right)^2}}$$

$$\leq\frac{1+\frac{2^{p-1}|c_p|}{H(1)\prod_{k=1}^{n}|b_k|}}{1-\frac{2^{p-1}|c_p|}{H(1)\prod_{k=1}^{n}|b_k|}}\left\{\frac{|H'(0)|}{H(1)}-p-\sum_{k=1}^{n}\frac{1-|b_k|^2}{|1+b_k|^2}\right\}$$

and

$$\frac{2\left(H(1)\prod_{k=1}^{n}|b_k|-\left(2^{p-1}|c_p|\right)\right)^2}{\left(H(1)\prod_{k=1}^{n}|b_k|\right)^2-\left(2^{p-1}|c_p|\right)^2+2^{p-1}H(1)\prod_{k=1}^{n}|b_k|\left|2c_{p+1}+c_p\left(p+\sum_{k=1}^{n}\frac{1-|b_k|^2}{b_k}\right)\right|}$$

$$\leq\frac{|H'(0)|}{H(1)}-p-\sum_{k=1}^{n}\frac{1-|b_k|^2}{|1+b_k|^2}.$$

Since

$$\frac{1-|b_k|^2}{b_k}=\frac{1-\left|\frac{s_k-1}{s_k+1}\right|^2}{\frac{s_k-1}{s_k+1}}=\frac{4\Re s_k}{|s_k|^2+2i\Im s_k-1}$$

and

$$\frac{1-|b_k|^2}{|1+b_k|^2}=\frac{1-\left|\frac{s_k-1}{s_k+1}\right|^2}{\left|1+\frac{s_k-1}{s_k+1}\right|^2}=\frac{\frac{4\Re s_k}{|s_k+1|^2}}{\frac{4|s_k|^2}{|s_k+1|^2}}=\frac{\Re s_k}{|s_k|^2}$$

hence we get inequality (2.2). Now we shall show that inequality (2.2) is sharp. Let

$$H\left(\frac{1+z}{1-z}\right)=\frac{1-z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}{1+z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}H(1)$$

$$=\left(-1+\frac{2}{1+z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}\right)H(1).$$

Then

$$\frac{2}{(1-z)^2}H'\left(\frac{1+z}{1-z}\right)$$

$$=\frac{-2\left((p+1)z^p\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}+z^{p+1}\sum_{k=1}^{n}\frac{1-|b_k|^2}{\left(1-\overline{b_k}z\right)^2}\prod_{\substack{m=1\\k\neq m}}^{n}\frac{z-b_m}{1-\overline{b_m}z}\right)}{\left(1+z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}\right)^2}H(1)$$

$$=\frac{-2z^p\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}\left(p+1+z\sum_{k=1}^{n}\frac{1-|b_k|^2}{\left(1-\overline{b_k}z\right)}\frac{1}{z-b_k}\right)}{\left(1+z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}\right)^2}H(1)$$

and for $z=-1$

$$H'(0)=2\frac{-2(-1)^p\prod_{k=1}^{n}\frac{-1-b_k}{1+\overline{b_k}}\left(p+1-\sum_{k=1}^{n}\frac{1-|b_k|^2}{\left(1+\overline{b_k}\right)}\frac{1}{-1-b_k}\right)}{\left(1+(-1)^{p+1}\prod_{k=1}^{n}\frac{-1-b_k}{1+\overline{b_k}}\right)^2}H(1)$$

$$=4\frac{(-1)^p\prod_{k=1}^{n}\frac{1+b_k}{1+\overline{b_k}}\left(p+1+\sum_{k=1}^{n}\frac{1-|b_k|^2}{\left(1+\overline{b_k}\right)}\frac{1}{1+b_k}\right)}{\left(1-(-1)^{p+1}\prod_{k=1}^{n}\frac{1+b_k}{1+\overline{b_k}}\right)^2}H(1)$$

and since $b_1,b_2,\ldots,b_n$ are positive real numbers, we have

$$H'(0)=4\frac{(-1)^p\left(p+1+\sum_{k=1}^{n}\frac{1-b_k^2}{(1+b_k)^2}\right)}{(1-(-1)^{p+1})^2}H(1).$$

Also, for $p=2,4,..,n$, we obtain

$$|H^{'}(0)| = H(1)\left(p+1+\sum_{k=1}^{n}\frac{1-b_k}{1+b_k}\right).$$

Since $b_k = \frac{s_k - 1}{s_k + 1}$, we get

$$|H^{'}(0)| = H(1)\left(p+1+\sum_{k=1}^{n}\frac{1}{s_k}\right).$$

On the other hand, we obtain

$$H(1) + c_p\frac{2^p z^p}{(1-z)^p} + c_{p+1}\frac{2^{p+1}z^{p+1}}{(1-z)^{p+1}} + \dots$$

$$= \left(1 - \frac{2z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}{1+z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}\right)H(1),$$

$$c_p\frac{2^p z^p}{(1-z)^p} + c_{p+1}\frac{2^{p+1}z^{p+1}}{(1-z)^{p+1}} + \dots = -\frac{2z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}{1+z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}H(1)$$

and

$$c_p\frac{2^p}{(1-z)^p} + c_{p+1}\frac{2^{p+1}z}{(1-z)^{p+1}} + \dots = -\frac{2z\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}{1+z^{p+1}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}H(1).$$

Passing to limit in the last equality yields $c_p = 0$. Similarly, using straightforward calculations, we get

$$c_{p+1} = \frac{1}{2^p}\prod_{k=1}^{n} b_k = \frac{1}{2^p}\prod_{k=1}^{n}\frac{s_k - 1}{s_k + 1}.$$

Therefore, (2.2) holds.

The results obtained in the presented theorems can be generalized by examining Schwarz lemma at the boundary for distinct zeros other than $s = 1$, such as s_1, s_2, s_3,..., s_k.

A similar analysis which was carried out for Theorem 1 can also be performed for Theorem 2. Using the obtained extremal function given in Theorem 2, following generic transfer function is obtained:

$$H_2(s) = \frac{b_1 s^{p+n} + b_2 s^{p+n-2} + \dots + b_{\frac{p+n+1}{2}}s}{a_1 s^{p+n+1} + a_2 s^{p+n-1} + \dots + a_{\frac{p+n+1}{2}+1}}, \quad p = 2,4,6,\dots,n.$$

where for $H(1)$ is taken as equal to 1. The related block diagram for H_2 is given in Fig. 6. In the figure, $G_1(s)$ and $G_2(s)$ are given as $G_1(s) = b^T s$ and $G_2(s) = s\frac{a^T s}{b^T s}$, respectively, where

$$a = \left[a_1\ a_2\ \dots\ a_{\frac{p+n+1}{2}}\right]^T, b = \left[b_1\ b_2\ \dots\ b_{\frac{p+n+1}{2}}\right]^T,$$

and

$$s = [s^{p+n}\ s^{p+n-2}\ \dots\ s]^T.$$

As an example, for $p = 2$ and $p = 4$ with $n = 1$, the obtained transfer functions are given, respectively, as follows:

$$H_{p=2}(s) = \frac{(s_1+3)s^3 + (1+3s_1)s}{s^4 + (3+3s_1)s^2 + s_1}$$

and

$$H_{p=4}(s) = \frac{(5+s_1)s^5 + (10+10s_1)s^3 + (1+5s_1)s}{s^6 + (10+5s_1)s^4 + (5+10s_1)s^2 + s_1}.$$

If we assume that $s_1 = 1$, these transfer functions become

$$H_{p=2}(s) = \frac{4s^3 + 4s}{s^4 + 6s^2 + 1}$$

and

$$H_{p=4}(s) = \frac{6s^5 + 20s^3 + 6s}{s^6 + 15s^4 + 15s^2 + 1}.$$

For these transfer functions, corresponding root-locus diagrams are given in Figs. 7 and 8, respectively. According to these figures, considered transfer functions define marginally stable systems as in Theorem 1. Also, the figures show that both transfer functions have a zero at infinity.

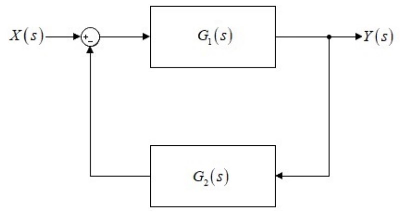

Fig. 6. Block diagram representation of the transfer function

$$H_2(s) = \frac{Y(s)}{X(s)} = \frac{b_1 s^{p+n} + b_2 s^{p+n-2} + \dots + b_{\frac{p+n+1}{2}}s}{a_1 s^{p+n+1} + a_2 s^{p+n-1} + \dots + a_{\frac{p+n+1}{2}+1}}, p = 2,4,6,\dots,n.$$

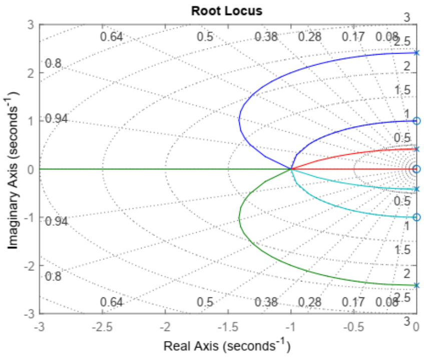

Fig. 7. Root-locus diagram for $H_{p=2}(s) = \frac{4s^3+4s}{s^4+6s^2+1}$.

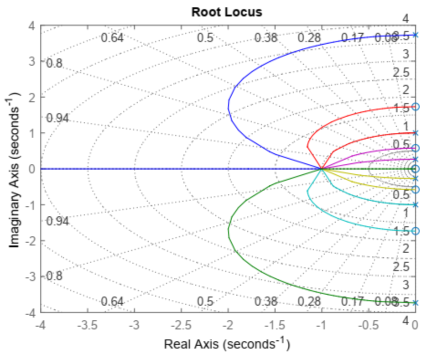

Fig. 8. Root-locus diagram for $H_{p=4}(s) = \frac{6s^5+20s^3+6s}{s^6+15s^4+15s^2+1}$.

4. Conclusions

In this paper, a boundary analysis for transfer functions of control theory has been carried out using Schwarz lemma. Two theorems are presented with their proofs by assuming that $H(s)$ is analytic at the origin with $H(0) = 0$. In these two theorems, lower boundaries for $|H^{'}(0)|$ have been obtained. The obtained inequalities used in sharpness analysis to determine transfer functions. Two unique transfer functions, $H_1(s)$ and $H_2(s)$, have been obtained for two unique theorems. It is worth to note that here the obtained transfer functions are not arbitrary but they are the intuitive results of the proposed theorems. Therefore, these functions have also been investigated in terms of root-locus graphics. According to observed results, it is possible to conclude that marginally stable systems are obtained using the results of the proposed theorems.

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